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Generalized complex geometry, generalized branes and the Hitchin sigma model

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Abstract: Hitchin's generalized complex geometry has been shown to be relevant in compactifications of superstring theory with fluxes and is expected to lead to a deeper understanding of mirror symmetry. Gualtieri's notion of generalized complex submanifold seems to be a natural candidate for the description of branes in this context. Recently, we introduced a Batalin-Vilkovisky field theoretic realization of generalized complex geometry, the Hitchin sigma model, extending the well known Poisson sigma model. In this paper, exploiting Gualtieri's formalism, we incorporate branes into the model. A detailed study of the boundary conditions obeyed by the world sheet fields is provided. Finally, it is found that, when branes are present, the classical Batalin-Vilkovisky cohomology contains an extra sector that is related non trivially to a novel cohomology associated with the branes as generalized complex submanifolds.

Keywords: Sigma Models, Topological Field Theories, BRST Symmetry, Differential and Algebraic Geometry.

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Contents

1. Introduction

Mirror symmetry is a duality relating compactifications of type IIA and type IIB superstring theory, which yield the same four-dimensional low energy effective theory. It has played an important role in the study of Calabi-Yau compactifications for both its theoretical implications and practical usefulness. Recently, more general compactifications allowing for non Ricci-flat metrics and NSNS and RR fluxes have been object of intense inquiry. The natural question arises about whether mirror symmetry generalizes to this broader class of compactifications and, if so, which its properties are. This program was outlined originally in refs. [1, 2] and was subsequently implemented with an increasing degree of generality in a series of papers [3]–[5]. These studies indicate that mirror symmetry can be defined for a class of manifolds with SU(3) structure. In spite of these advancements, certain aspects of mirror symmetry remain mysterious.

In 2002, Hitchin formulated the notion of generalized complex geometry, which at the same time unifies and extends the customary notions of complex and symplectic geometry and incorporates a natural generalization of Calabi-Yau geometry [6]. Hitchin's ideas were developed by Gualtieri [7], who also worked out the theory of generalized Kaehler geometry. The SU(3) structure manifolds considered in [4, 5] are generalized Calabi-Yau manifolds as defined by Hitchin. This indicates that generalized complex geometry may provide the mathematical set up appropriate for the study of mirror symmetry for general flux compactifications.

Type II superstring Calabi-Yau compactifications are described by $(2,2)$ superconformal sigma models with Calabi-Yau target manifolds. These field theories are however rather complicated and, so, they are difficult to study. In 1988, Witten showed that a $(2,2)$

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supersymmetric sigma model on a Calabi-Yau space could be twisted in two different ways, to give the so called A and B topological sigma models [8]. Unlike the original untwisted sigma model, the topological models are soluble: the calculation of observables can be reduced to classical problems of geometry. For this reason, the topological models constitute an ideal field theoretic ground for the study of mirror symmetry.

Mirror symmetry relates the A and B models with mirror target manifolds [9]. The A and B models depend only on the symplectic and complex geometry of the target manifold, respectively. Therefore, generalized complex geometry, which unifies these two types of geometry, may provide a natural mathematical framework for a unified understanding of them. It is also conceivable that topological sigma models with generalized Kaehler targets may exhibit the form of generalized mirror symmetry encountered in flux compactifications of superstring theory [10]–[15].

D-branes are extended solitonic objects of the superstring spectrum, which are expected to play an important role in the ultimate non perturbative understanding of superstring physics. They appear as hypersurfaces in space-time where the ends of open strings are constrained to lie. D-branes appear in a $(2, 2)$ superconformal sigma model as certain boundary conditions for the world sheet fields. Topological branes appear in the associated topological sigma models again as boundary conditions. The branes of the A and B model are called A- and B-branes, respectively. Expectedly, mirror symmetry exchanges the sets of A-branes and B-branes of mirror manifolds. This constitutes a major motivation for their study.

In 1994, Kontsevich formulated the homological mirror symmetry conjecture [16], which interprets mirror symmetry as the equivalence of two triangulated categories: the bounded derived category of coherent sheaves and the derived Fukaya category of graded lagrangian submanifolds carrying flat vector bundles. After the emergence of D-branes in superstring theory, it was argued that the topological B-branes formed a category which could be identified with the derived category of coherent sheaves [17, 18]. Since mirror symmetry exchanges A-branes and B-branes, one would expect that the topological A-branes also form a category and that this category could be identified with the derived Fukaya category. This belief was supported by Witten's original work [19], where A-branes appeared as lagrangian submanifolds, and by the analysis of ref. [20], which showed that ghost number anomaly cancellation required these lagrangian submanifolds to be graded. However, other studies indicated that there could be A-branes which were not lagrangian submanifolds [21]. The careful analysis of [22] showed that a class of coisotropic submanifolds carrying non trivial line bundles could also serve as A-branes, at least at the classical level. This finding suggests that the category of A-branes should be an appropriate extension of the derived Fukaya category including the coisotropic branes.

For reasons explained above, it is expected that generalized complex geometry may furnish the appropriate framework for the understanding of mirror symmetry also in the presence of branes. In this case, the notion of generalized complex submanifold formulated by Gualtieri in [7] should play an important role. In fact, it includes as particular cases all known examples of topological branes, including the coisotropic ones. However, to the best of our knowledge, not much has been tried in this direction so far [10, 15, 23].

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Any attempt to understand mirror symmetry in the light of generalized complex geometry unavoidably must go through some sigma model realization of this latter. Several such realizations have been proposed so far [10, 13] and [23]–[32]. They have remarkable properties and are also interesting in their own. This paper is a further step along this line of development, as we shall describe next.

To a varying degree, the sigma model realizations of generalized complex geometry are all related to the well-known Poisson sigma model [33, 34]. This is not surprising, in view of the fact that a generalized complex manifold is also a Poisson manifold. In [31], adapting and generalizing the formulation of the Poisson sigma model of ref. [35], based on the Batalin-Vilkovisky quantization algorithm [36, 37], we formulated a new realization called Hitchin sigma model. We showed that the algebraic properties and integrability conditions of the target space generalized almost complex structure are sufficient, non necessary conditions for the fulfillment of the Batalin-Vilkovisky classical master equation $(S, S) = 0$. Further, a non trivial relation between a sector of the classical Batalin-Vilkovisky cohomology, on one hand, and a generalized Dolbeault cohomology and the cohomology of the generalized deformation complex, on the other, was found.

In this paper, we continue the study of Hitchin sigma model on a generalized complex manifold and show how generalized branes can be incorporated into it. We argue that branes are aptly described by Gualtieri's theory of generalized complex submanifolds [7]. In the presence of a brane, the world sheet has a non empty boundary. The fields therefore must obey appropriate boundary conditions, which are determined and analyzed in detail in the paper. We show that the algebraic properties and integrability conditions of the target space generalized almost complex structure as well as the algebraic properties of the brane as generalized complex submanifold are sufficient, non necessary conditions for the fulfillment of the Batalin-Vilkovisky classical master equation $(S, S) = 0$. The compatibility of the boundary conditions with the so called b symmetry and with the overall classical Batalin-Vilkovisky cohomological structure is ascertained. Further, we find that, when branes are present, the Batalin-Vilkovisky cohomology contains an extra sector that is related non trivially to an hitherto unknown cohomology associated with the branes as generalized complex submanifolds. Finally, we compare our approach with the Alexandrov-Kontsevich-Schwartz-Zaboronsky formulation of the Poisson sigma model without and with branes worked out in refs. [38, 39].

The plan of this paper is as follows. In section 2, we review the basic notions of twisted generalized complex geometry. In section 3, we illustrate the theory of generalized complex submanifolds and show its use for the description of generalized branes. In section 4, we introduce the 2-dimensional de Rham superfield formalism for world sheets with boundary. In section 5, we review the twisted Hitchin sigma model in the absence of branes and then show how branes can be incorporated by imposing suitable boundary conditions on the fields. In section 6, we analyze the compatibility of the boundary conditions with the Batalin-Vilkovisky cohomological structure when branes are present and describe in detail the relation between the Batalin-Vilkovisky cohomology and the generalized complex submanifold cohomology of the branes. Finally, in section 7, we compare our results with related works on the Poisson sigma model with coisotropic branes.

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2. Generalized complex geometry

The notion of generalized complex structure was introduced by Hitchin in [6] and developed by Gualtieri [7] in his thesis. It encompasses the usual notions of complex and symplectic structure as special cases. It is the complex counterpart of the notion of Dirac structure, introduced by Courant and Weinstein, which unifies Poisson and symplectic geometry [40, 41]. In this section, we review the basic definitions and results of generalized complex geometry used in the sequel of the paper.

Let M be a manifold. In what follows, M will always be of even dimension d, since the basic structures of generalized complex geometry exist only in this case.

The constructions of generalized complex geometry are all based on the vector bundle $TM \oplus T^*M$. A generic section $X+\xi \in C^\infty(TM \oplus T^*M)$ of this bundle is the direct sum of a vector field $X \in C^\infty(TM)$ and a 1-form $\xi \in C^\infty(T^*M)$.

$TM \oplus T^*M$ is equipped with a natural indefinite metric of signature (d,d) defined by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(i_X\eta + i_Y\xi)\,, \tag{2.1}$$

for $X+\xi, Y+\eta \in C^\infty(TM\oplus T^*M)$, where i_V denotes contraction with respect to a vector field V. This metric has a large isometry group. This contains the full diffeomorphism group of M, acting by pull-back. It also contains the following distinguished isometries, called b transforms, defined by

$$\exp(b)(X+\xi) = X+\xi+i_X b\,, \tag{2.2}$$

where $b \in C^\infty(\wedge^2 T^*M)$ is a 2-form. As it turns out, b transformation is the most basic symmetry of generalized complex geometry.

An H field is a closed 3-form $H \in C^\infty(\wedge^3 T^*M)$:

$$d_M H = 0\,, \tag{2.3}$$

where d_M is the exterior differential of M. [1] H measures the so called twisting. The pair (M,H) is called a twisted manifold. (M,H) is characterized by the cohomology class $[H] \in H^3(M,\mathbb{R})$. By definition, b transformation shifts H by the exact 3-form $-d_M b$:

$$H' = H - d_M b\,. \tag{2.4}$$

So, the cohomology class $[H]$ is invariant.

On a twisted manifold (M,H), there is a natural bilinear pairing defined on $C^\infty(TM\oplus T^*M)$ extending the customary Lie pairing on $C^\infty(TM)$, called H twisted Courant brackets [40, 41]. It is given by the expression

$$[X+\xi, Y+\eta]_H = [X,Y] + l_X\eta - l_Y\xi - \frac{1}{2}d_M(i_X\eta - i_Y\xi) + i_X i_Y H\,, \tag{2.5}$$

with $X+\xi, Y+\eta \in C^\infty(TM\oplus T^*M)$, where l_V denotes Lie derivation with respect to a vector field V. The pairing is antisymmetric, but it fails to satisfy the Jacobi identity. However, remarkably, the Jacobi identity is satisfied when restricting to sections $X+$

[1] The sign convention of the H field used in this paper is opposite to that of ref. [7].

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$\xi, Y+\eta \in C^\infty(L)$, where L is a subbundle of $TM \oplus T^*M$ isotropic with respect to $\langle\,,\rangle$ and involutive (closed) under $[\,,]_H$. The brackets $[\,,]_H$ are covariant under the action of the diffeomorphism group. They are also covariant under b transform,

$$[\exp(b)(X+\xi), \exp(b)(Y+\eta)]_H = \exp(b)[X+\xi, Y+\eta]_{H'}\,, \tag{2.6}$$

where H' is the b transform of H given by (2.4).

A generalized almost complex structure $\mathcal{J}$ is a section of $C^\infty(\mathrm{End}(TM \oplus T^*M))$, which is an isometry of the metric $\langle\,,\rangle$ and satisfies

$$\mathcal{J}^2 = -1\,. \tag{2.7}$$

The pair $(M, \mathcal{J})$ is called a generalized almost complex manifold. The group of isometries of $\langle\,,\rangle$ acts on $\mathcal{J}$ by conjugation. In particular, the b transform of $\mathcal{J}$ is given by

$$\mathcal{J}' = \exp(-b)\mathcal{J}\exp(b)\,. \tag{2.8}$$

If M is equipped with an H field and a generalized almost complex structure $\mathcal{J}$, the triple $(M, H, \mathcal{J})$ is called a twisted generalized almost complex manifold. The generalized almost complex structure $\mathcal{J}$ is H integrable if its $\pm\sqrt{-1}$ eigenbundles are involutive with respect to the twisted Courant brackets $[\,,]_H$.[2] In that case, $\mathcal{J}$ is called an H twisted generalized complex structure. The triple $(M, H, \mathcal{J})$ is then called a twisted generalized complex manifold. It can be shown that this condition is equivalent to the vanishing of the appropriate H twisted generalized Nijenhuis tensor

$$N_H(X+\xi, Y+\eta) = 0\,, \tag{2.9}$$

for all $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$, where

$$\begin{aligned} N_H(X+\xi, Y+\eta) = {} & [X+\xi, Y+\eta]_H + \mathcal{J}[\mathcal{J}(X+\xi), Y+\eta]_H + \\ & + \mathcal{J}[X+\xi, \mathcal{J}(Y+\eta)]_H - [\mathcal{J}(X+\xi), \mathcal{J}(Y+\eta)]_H\,. \end{aligned} \tag{2.10}$$

H integrability is preserved by b transformation: if $\mathcal{J}$ is an H twisted generalized complex structure and H' and $\mathcal{J}'$ are the b transform of H and $\mathcal{J}$, respectively, then $\mathcal{J}'$ is an H' twisted generalized complex structure.

In the untwisted formulation of generalized complex geometry, the H field vanishes throughout. To preserve the condition $H = 0$, b transformation must be restricted. By (2.4), it follows that only closed b fields are allowed, $d_M b = 0$. This restriction is not necessary in the twisted case, which is the one we deal with mostly in this paper.

In practice, it is convenient to decompose a generalized almost complex structure $\mathcal{J}$ in block matrix form as follows

$$\mathcal{J} = \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix}, \tag{2.11}$$

where $J \in C^\infty(TM \otimes T^*M)$, $P \in C^\infty(\wedge^2 TM)$, $Q \in C^\infty(\wedge^2 T^*M)$ and express all the properties of $\mathcal{J}$, in particular its integrability, in terms of the blocks J, P, Q.

For later use, we write in explicit tensor notation the conditions obeyed by the fields H, J, P, Q.

[2]The $\pm\sqrt{-1}$ eigenbundles of $\mathcal{J}$ are complex and, thus, their analysis requires complexifying $TM \oplus T^*M$ leading to $(TM \oplus T^*M) \otimes \mathbb{C}$.

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An H field satisfies the closedness equation

$$\partial_a H_{bcd} - \partial_b H_{acd} + \partial_c H_{abd} - \partial_d H_{abc} = 0\,, \tag{2.12}$$

(cf. eq. (2.3)). Under b transform, we have

$$H'_{abc} = H_{abc} - \big(\partial_a b_{bc} + \partial_b b_{ca} + \partial_c b_{ab}\big), \tag{2.13}$$

(cf. eq. (2.4)).

If $\mathcal{J}$ is a generalized almost complex structure, the tensors J, P, Q satisfy

$$P^{ab} + P^{ba} = 0\,, \tag{2.14a}$$
$$Q_{ab} + Q_{ba} = 0\,, \tag{2.14b}$$

$$J^a{}_c J^c{}_b + P^{ac} Q_{cb} + \delta^a{}_b = 0\,, \tag{2.15a}$$
$$J^a{}_c P^{cb} + J^b{}_c P^{ca} = 0\,, \tag{2.15b}$$
$$Q_{ac} J^c{}_b + Q_{bc} J^c{}_a = 0\,, \tag{2.15c}$$

on account of (2.7). Upon using (2.2), (2.8), we find that, under b transform,

$$P'^{ab} = P^{ab}\,, \tag{2.16a}$$
$$J'^a{}_b = J^a{}_b - P^{ac} b_{cb}\,, \tag{2.16b}$$
$$Q'_{ab} = Q_{ab} + b_{ac} J^c{}_b - b_{bc} J^c{}_a + P^{cd} b_{ca} b_{db}\,. \tag{2.16c}$$

The H integrability condition (2.9) of a generalized almost complex structure $\mathcal{J}$ can be cast in the form of a set of four tensorial equations

$$A_H{}^{abc} = 0\,, \tag{2.17a}$$
$$B_{Ha}{}^{bc} = 0\,, \tag{2.17b}$$
$$C_{Hab}{}^{c} = 0\,, \tag{2.17c}$$
$$D_{Habc} = 0\,, \tag{2.17d}$$

where A_H, B_H, C_H, D_H are the tensors defined by

$$A_H{}^{abc} = P^{ad}\partial_d P^{bc} + P^{bd}\partial_d P^{ca} + P^{cd}\partial_d P^{ab}\,, \tag{2.18a}$$

$$\begin{aligned} B_{Ha}{}^{bc} = {}& J^d{}_a \partial_d P^{bc} + P^{bd}(\partial_a J^c{}_d - \partial_d J^c{}_a) - P^{cd}(\partial_a J^b{}_d - \partial_d J^b{}_a) - \\ & - \partial_a (J^b{}_d P^{dc}) + P^{bd} P^{ce} H_{ade}\,, \end{aligned} \tag{2.18b}$$

$$\begin{aligned} C_{Hab}{}^{c} = {}& J^d{}_a \partial_d J^c{}_b - J^d{}_b \partial_d J^c{}_a - J^c{}_d \partial_a J^d{}_b + J^c{}_d \partial_b J^d{}_a + \\ & + P^{cd}(\partial_d Q_{ab} + \partial_a Q_{bd} + \partial_b Q_{da}) - J^d{}_a P^{ce} H_{bde} + J^d{}_b P^{ce} H_{ade}\,, \end{aligned} \tag{2.18c}$$

$$\begin{aligned} D_{Habc} = {}& J^d{}_a(\partial_d Q_{bc} + \partial_b Q_{cd} + \partial_c Q_{db}) + J^d{}_b(\partial_d Q_{ca} + \partial_c Q_{ad} + \partial_a Q_{dc}) + \\ & + J^d{}_c(\partial_d Q_{ab} + \partial_a Q_{bd} + \partial_b Q_{da}) - \partial_a (Q_{bd} J^d{}_c) - \partial_b (Q_{cd} J^d{}_a) - \partial_c (Q_{ad} J^d{}_b) - \\ & - H_{abc} + J^d{}_a J^e{}_b H_{cde} + J^d{}_b J^e{}_c H_{ade} + J^d{}_c J^e{}_a H_{bde}\,. \end{aligned} \tag{2.18d}$$

The above expressions were first derived in a different but equivalent form in [25] and subsequently in the form given here in [31].

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One of the most interesting features of generalized complex geometry is its capability for a unified treatment of complex and symplectic geometry, as we show below. However, as noticed by Hitchin, these geometries do not exhaust the scope of generalized complex geometry. In fact, there are manifolds which cannot support any complex or symplectic structures, but do admit generalized complex structures [6]. These facts explain the reason why Hitchin's construction is interesting and worthwhile pursuing and not simply an elegant repackaging of known notions.

Complex geometry can be formulated in terms of generalized almost complex structures $\mathcal{J}$ of the form

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \tag{2.19}$$

where J is an ordinary almost complex structures i.e $J^a{}_c J^c{}_b = -\delta^a{}_b$. $\mathcal{J}$ satisfies ((2.17a)–(2.17d)) if J, H satisfy

$$J^d{}_a \partial_d J^c{}_b - J^d{}_b \partial_d J^c{}_a - J^c{}_d \partial_a J^d{}_b + J^c{}_d \partial_b J^d{}_a = 0\,, \tag{2.20}$$

$$H_{abc} - J^d{}_a J^e{}_b H_{cde} - J^d{}_b J^e{}_c H_{ade} - J^d{}_c J^e{}_a H_{bde} = 0\,. \tag{2.21}$$

Eq. (2.20) is nothing but the statement of the vanishing of the Nijenhuis tensor of J and, so, it implies that J is a complex structure. Eq. (2.21) states in turn the 3–form H is of type $(2,1)+(1,2)$ with respect to J.

Similarly, symplectic geometry can be formulated in terms of generalized almost complex structures $\mathcal{J}$ of the form

$$\mathcal{J} = \begin{pmatrix} 0 & -Q^{-1} \\ Q & 0 \end{pmatrix}, \tag{2.22}$$

where Q is pointwise non singular a 2–form. $\mathcal{J}$ satisfies (2.17a)–(2.17d) if Q, H satisfy

$$\partial_a Q_{bc} + \partial_b Q_{ca} + \partial_c Q_{ab} = 0\,, \tag{2.23}$$

$$H_{abc} = 0\,. \tag{2.24}$$

Eq. (2.23) states that the 2–form Q is closed and, so, it implies that Q is a symplectic structure. By eq. (2.24), the H field necessarily vanishes in the symplectic case.

In the Hitchin sigma model studied in [31], the action S contains a topological Wess-Zumino term defined up to the periods of the closed 3–form H. In the quantum path integral, so, the weight $\exp(\sqrt{-1}S)$ is unambiguously defined only if $H/2\pi$ has integer periods. Therefore, the cohomology class $[H/2\pi]$ belongs to the image of $H^3(M,\mathbb{Z})$ in $H^3(M,\mathbb{R})$. This case is particular important for its relation to gerbes.[3] In this context, the b tansforms with b a closed 2-form such that $[b/2\pi]$ is contained in the image of $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$ represent the gerbe generalization of gauge transformations.

In [43], it was shown that a sigma model on a manifold M with NSNS background H has $(2,2)$ supersymmetry if the twisted manifold (M,H) is "Kaehler with torsion". This means that M is equipped with a riemannian metric g and two generally different complex

[3]See [42] for background material about this topic.

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structures $J_\pm$ such that g is hermitean with respect to both $J_\pm$ and that $J_\pm$ are parallel with respect to two different metric connections $\nabla_\pm$ with torsion proportional to $\pm H$. The presence of torsion implies that the geometry is not Kaehler. As shown in [7], these geometrical data can be assembled in a pair of commuting H twisted generalized complex structures $\mathcal{J}_i$, $i = 1, 2$, describing a generalized Kaehler geometry.

The $(2,2)$ supersymmetric sigma model has been studied mostly for $H = 0$ and $J_+ = J_-$, when M is a true Kaehler manifold. In this case, the generalized complex structures $\mathcal{J}_1$ $\mathcal{J}_2$ are of the special form (2.19), (2.22) and encode the complex and symplectic geometry of M, respectively. The associated A and B topological sigma models depend on only one of these, depending on the topological twisting used [8]: the A model depends only on $\mathcal{J}_2$, the B model only on $\mathcal{J}_1$. A topological sigma model for generalized Kaehler geometry has been proposed in [13]. The Hitchin sigma model [31], which is the main topic of this paper, can be defined for a general twisted generalized complex structure $\mathcal{J}$ not necessarily of the form of those appearing in generalized Kaehler geometry.

3. Generalized complex submanifolds and branes

According to [7], a generalized submanifold of a twisted manifold (M, H) (cf. section 2) is a pair (W, F), where W is a submanifold of M and $F \in C^\infty(\wedge^2 T^*W)$ is such that

$$\iota_W{}^* H = d_W F\,, \tag{3.1}$$

where $\iota_W : W \to M$ is the natural injection and d_W is the differential of W. Note that, by (2.3)) and (3.1), the pair (H, F) is a representative of a relative cohomology class in $H^3(M, W, \mathbb{R})$. By (2.4), (3.1), F transforms non trivially under a b transform, viz

$$F' = F - \iota_W{}^* b\,. \tag{3.2}$$

where $b \in C^\infty(\wedge^2 T^*M)$. In this way, if (W, F) is a generalized submanifold of (M, H), then (W, F') is a generalized submanifold of (M, H'), where H', F' are the b transforms of H, F given by (2.4), (3.2).

The generalized tangent bundle $\mathcal{T}^F W$ of (W, F) is the subbundle of $TM \oplus T^*M|_W$ spanned by the restriction to W of those sections $X + \xi \in C^\infty(TM \oplus T^*M)$ such that $X|_W \in C^\infty(TW)$ and that

$$\iota_W{}^* \xi = i_{X|_W} F\,. \tag{3.3}$$

(Here, TW is viewed as a subbundle of $TM|_W$.) It is easy to see that b transform acts naturally on $\mathcal{T}^F W$. Indeed, one has $\exp(b)\mathcal{T}^{F'} W = \mathcal{T}^F W$. In fact, this is the main reason why $\mathcal{T}^F W$ is defined the way indicated above.

Suppose $(M, H, \mathcal{J})$ is a twisted generalized almost complex manifold (cf. section 2). A generalized submanifold (W, F) of (M, H) is a generalized almost complex submanifold of $(M, H, \mathcal{J})$, if $\mathcal{T}^F W$ is stable under the action of $\mathcal{J}$ seen here as a section of $C^\infty(\mathrm{End}(TM \oplus$

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T^*M)). When $\mathcal{J}$ is H integrable and, therefore, $(M, H, \mathcal{J})$ is a twisted generalized complex manifold, we call (W, F) a generalized complex submanifold of $(M, H, \mathcal{J})$.[4] This notion is covariant under b transformation: if (W, F) is a generalized almost complex submanifold of $(M, H, \mathcal{J})$, then (W, F') is a generalized almost complex submanifold of $(M, H', \mathcal{J}')$, where H', $\mathcal{J}'$, F' are the b transforms of H, $\mathcal{J}$, F given by (2.4), (2.8), (3.2).

As is well known, at each point of W, there are coordinates of M $t^a = (t^i, t^\rho)$ such that, locally, the submanifold W is described by the equation $t^\rho = 0$ and coordinatized by the t^i. We call such coordinates adapted. Here, middle Latin indices i, $j, \dots$ take the values $1, \dots, \mathrm{dim} W$, while late Greek indices ρ, $\sigma, \dots$ take the values $1, \dots, d - \mathrm{dim} W$. The abstract geometrical notions outlined above are conveniently expressed in terms of adapted coordinates. The relations so obtained are adapted covariant, that is they have the same mathematical form for all choices of adapted coordinates. The tensor components H_{abc}, P^{ab}, $J^a{}_b$ Q_{ab} of the H field and the blocks P, J, Q of the generalized almost complex structure $\mathcal{J}$ (cf. eq. (2.11) entering in them are tacitly assumed to be restricted to W to avoid the cumbersome repetition of $\big|_W$. Since the F field is defined only on W anyway, no restriction of the components F_{ij} is involved.

Relation (3.1), connecting H and F reads simply

$$H_{ijk} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} \tag{3.4}$$

at W. Under a b transform, one has

$$F'_{ij} = F_{ij} - b_{ij}\,, \tag{3.5}$$

as follows from (3.2).

If $X + \xi \in C^\infty(TM \oplus T^*M)$ restricts to a section of the generalized tangent space $\mathcal{T}^F W$, then one has

$$X^\rho = 0\,, \tag{3.6a}$$

$$\xi_i + F_{ij} X^j = 0 \tag{3.6b}$$

at W, as follows from (3.3).

If $(M, H, \mathcal{J})$ is a twisted generalized almost complex manifold and (W, F) is a generalized almost complex submanifold of $(M, H, \mathcal{J})$, then, in the block representation (2.11) of $\mathcal{J}$, one has

$$P^{\rho\sigma} = 0\,, \tag{3.7a}$$

$$J^\rho{}_i - P^{\rho j} F_{ji} = 0\,, \tag{3.7b}$$

$$Q_{ij} - J^k{}_i F_{jk} + J^k{}_j F_{ik} + P^{kl} F_{ki} F_{lj} = 0 \tag{3.7c}$$

at W. These relations follow from (3.6a), (3.6b) and imposing the stability of $\mathcal{T}^F W$ under the action of $\mathcal{J}$. It is straightforward to verify that these conditions are compatible with H integrability conditions (2.17a)–(2.17d).

[4] A generalization of Gualtieri's definition of generalized complex submanifold can be found in ref. [44].

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Suppose that $\mathcal{J}$ is an H twisted generalized complex structure of the form (2.19). Then, J, H satisfy (2.20), (2.21) and, so, J is a complex structure and H is $(2,1)+(1,2)$ 3–form. In this case, eqs. (3.7a) –(3.7c) become simply

$$J^\rho{}_i = 0\,, \tag{3.8a}$$

$$J^k{}_i F_{kj} + J^k{}_j F_{ik} = 0 \tag{3.8b}$$

at W. Eq. (2.20), (3.8a) entails that the $J^i{}_j$ are the components of a complex structure J_W on W, which is therefore a complex submanifold of M. Eq. (3.8b) entails in turn that F is (1,1) 2–form of W.

Suppose that $\mathcal{J}$ is an H twisted generalized complex structure of the form (2.22). Then, Q, H satisfy (2.23), (2.24) and, so, Q is a symplectic structure and H vanishes. Then, eqs. (3.7a)–(3.7c) become simply

$$Q^{-1\rho\sigma} = 0\,, \tag{3.9a}$$

$$Q^{-1\rho j} F_{ji} = 0\,, \tag{3.9b}$$

$$Q_{ij} - Q^{-1kl} F_{ki} F_{lj} = 0 \tag{3.9c}$$

at W. Further, as $H = 0$, (3.4) furnishes

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0\,. \tag{3.10}$$

Eq. (3.9a) entails that Q^{-1} maps the conormal bundle N^*W into the tangent bundle TW of W. This means that, by definition, that W is coisotropic. Condition (3.10) entails that F is a closed 2–form. The remaining conditions (3.9b), (3.9c) do not have any simple geometrical interpretation. (See however [7] for an attempt in this direction.) The interpretation of the geometry of W simplifies if one adds by hand the condition

$$Q_{ij} = 0 \tag{3.11}$$

at W, in virtue of which (3.9b), (3.9c) are automatically fulfilled, as is easy to see. Eq. (3.11) entails that Q maps the tangent bundle TW into the conormal bundle N^*W of W. Hence, when (3.7a) and (3.11) are satisfied, the submanifold W is simultaneously isotropic and coisotropic, hence lagrangian. This means that, by definition, that W is isotropic. (3.11) further entails the vanishing of the F field

$$F_{ij} = 0\,. \tag{3.12}$$

In this paper, we shall provide fresh evidence in favor of the claim that, in a consistent sigma model on a twisted generalized complex manifold $(M, H, \mathcal{J})$, branes should be generalized complex submanifolds (W, F) [10, 15, 23]. We note that the formalism expounded above is suitable only for the description of non coincident branes. In the case, often considered in string theory, of stacks of overlapping branes, one would need a non abelian generalization of the above construction which is not yet available [19].

In the brane Hitchin sigma model studied in this paper, branes are generic generalized complex submanifolds. So, we call them generalized branes.

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The action S_W of the model contains a topological Wess-Zumino term defined up to the relative periods of the closed relative 3-form (H, F). In the quantum path integral, so, the weight $\exp(\sqrt{-1}S_W)$ is unambiguously defined only if $(H/2\pi, F/2\pi)$ has integer relative periods. Therefore, the relative cohomology class $[(H/2\pi, F/2\pi)]$ belongs to the image of $H^3(M, W, \mathbb{Z})$ in $H^3(M, W, \mathbb{R})$. This indicates that generalized branes support relative gerbes with connection and curving [42].

In the A and B topological sigma models with branes, the H field vanishes and the branes support abelian gauge fields of field strength F. The ends of open strings lie on the branes and carry Chan-Paton point charges, which couple to gauge fields of the branes.

As is well known, the consistent interaction of a point charge with a gauge field requires that the gauge field strength has quantized fluxes. Since the open strings carry Chan-Paton point charges coupling to gauge fields of the branes, the brane gauge curvature $F/2\pi$ has integer periods. It follows that the A- and B-branes always support line bundles with connections [19].

Since open strings are involved, the world sheets have non empty boundaries and the world sheet fields obey boundary conditions, whose nature depends on the topological twisting and restricts the type of submanifolds of the Kaehler target space branes can be [21]. In the A model, in its simplest form, the boundary conditions are such that the complex structure exchanges normal and tangent directions to the brane and, for this reason, an A-brane must be a lagrangian submanifold with respect to the Kaehler symplectic structure. In the B model, conversely, the boundary conditions make the complex structure preserve normal and tangent directions and, so, a B-brane must be a complex submanifold with respect to the Kaehler complex structure.

Further restrictions arise from requirement that the coupling of the open string Chan-Paton charges to the brane gauge fields is invariant under the nilpotent topological charge [19]. In A model, this implies that the field strength F vanishes, so that the underlying line bundle is flat. In the B model, conversely, this makes F be a $(1, 1)$ 2-form and the underlying line bundle holomorphic.

Other restrictions follow from the requirement of ghost number anomaly cancellation. In the A model, this requires that the branes are graded lagrangian submanifolds of vanishing Maslov class [20].

To summarize, the above qualitative discussion indicates that an A-brane is a lagrangian submanifold carrying a flat line bundle with flat connection and that a B-brane is a complex submanifold carrying a holomorphic line bundle with connection with $(1, 1)$ curvature. The careful analysis of Kapustin and Orlov [22] shows that, in the A model, a class of coisotropic submanifolds carrying non flat line bundles can also serve as A-branes, at least at the classical level. (See also the recent proposal of ref. [15].)

The above brief review of how branes show up in the A and B topological sigma models fits quite well Gualtieri's formalism of generalized complex submanifolds illustrated above. Indeed, A- and B-branes are generalized complex submanifolds of the types described in the paragraphs of eqs. (3.9a) and (3.8a), respectively, and, thus, also particular examples of generalized branes

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4. 2-dimensional de Rham superfields

In general, the fields of a 2-dimensional field theory are differential forms on a oriented 2-dimensional manifold Σ. They can be viewed as elements of the space $\mathrm{Fun}(\Pi T\Sigma)$ of functions on the parity reversed tangent bundle $\Pi T\Sigma$ of Σ, which we shall call de Rham superfields [31, 35]. More explicitly, we associate with the coordinates z^α of Σ Grassmann odd partners ζ^α with

$$\mathrm{deg} z^\alpha = 0\,, \qquad \mathrm{deg}\zeta^\alpha = 1\,. \tag{4.1}$$

ζ^α transforms as the differential of z^α under coordinate changes. A de Rham superfield $\psi(z,\zeta)$ is a triplet formed by a 0-, 1-, 2-form field $\psi^{(0)}(z)$, $\psi^{(1)}{}_\alpha(z)$, $\psi^{(2)}{}_{\alpha\beta}(z)$ set as

$$\psi(z,\zeta) = \psi^{(0)}(z) + \zeta^\alpha \psi^{(1)}{}_\alpha(z) + \frac{1}{2}\zeta^\alpha\zeta^\beta\psi^{(2)}{}_{\alpha\beta}(z)\,. \tag{4.2}$$

The forms $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ are called the de Rham components of ψ.

$\Pi T\Sigma$ is endowed with a natural differential d defined by

$$dz^\alpha = \zeta^\alpha\,, \qquad d\zeta^\alpha = 0\,. \tag{4.3}$$

In this way, the exterior differential d of Σ can be identified with the operator

$$d = \zeta^\alpha \frac{\partial}{\partial z^\alpha}\,. \tag{4.4}$$

The coordinate invariant integration measure of $\Pi T\Sigma$ is

$$\mu = \mathrm{d}z^1\mathrm{d}z^2\mathrm{d}\zeta^1\mathrm{d}\zeta^2\,. \tag{4.5}$$

Any de Rham superfield ψ can be integrated on $\Pi T\Sigma$ according to the prescription

$$\int_{\Pi T\Sigma} \mu\,\psi = \int_\Sigma \frac{1}{2}\mathrm{d}z^\alpha\mathrm{d}z^\beta\psi^{(2)}{}_{\alpha\beta}(z)\,. \tag{4.6}$$

Similarly, fields on the boundary $\partial\Sigma$ can be viewed as elements of the space $\mathrm{Fun}(\Pi T\,\partial\Sigma)$ of functions on the parity reversed tangent bundle $\Pi T\partial\Sigma$ of $\partial\Sigma$, which we shall call boundary de Rham superfields. Again, we associate with the coordinates s of $\partial\Sigma$ Grassmann odd partners ς with

$$\mathrm{deg} s = 0\,, \qquad \mathrm{deg}\varsigma = 1\,. \tag{4.7}$$

ς transforms as the differential of s under coordinate changes. A generic boundary de Rham superfield $\chi(s,\varsigma)$ is a doublet formed by a 0-, 1-form field $\chi^{(0)}(s)$, $\chi^{(1)}{}_s(s)$, organized as

$$\chi(s,\varsigma) = \chi^{(0)}(s) + \varsigma\chi^{(1)}{}_s(s)\,. \tag{4.8}$$

The forms $\chi^{(0)}$, $\chi^{(1)}$ are called the boundary de Rham components of χ.

$\Pi T\partial\Sigma$ is endowed with a natural differential d_∂ defined by

$$d_\partial s = \varsigma\,, \qquad d_\partial\varsigma = 0\,. \tag{4.9}$$

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The exterior differential d_∂ of $\partial\Sigma$ can be identified in this way with the operator

$$d_\partial = \varsigma \frac{d}{ds} \,. \tag{4.10}$$

The coordinate invariant integration measure of $\Pi T\partial\Sigma$ is

$$\mu_\partial = \mathrm{d}s\mathrm{d}\varsigma \,. \tag{4.11}$$

Any de Rham superfield χ can be integrated on $\Pi T\partial\Sigma$ according to the prescription

$$\oint_{\Pi T\partial\Sigma} \mu_\partial \, \chi = \oint_{\partial\Sigma} \mathrm{d}s \chi^{(1)}{}_s(s) \,. \tag{4.12}$$

Let $\iota_\partial : \partial\Sigma \to \Sigma$ be the natural injection. If ψ is a de Rham superfield of Σ, then

$$\iota_\partial{}^*\psi(s,\varsigma) = \psi^{(0)}(\iota_\partial(s)) + \varsigma \frac{d\iota_\partial{}^\alpha}{ds}(s)\psi^{(1)}{}_\alpha(\iota_\partial(s)) \tag{4.13}$$

is a boundary de Rham superfield, the pull–back of ψ. By Stokes' theorem,

$$\int_{\Pi T\Sigma} \mu \, d\psi = \oint_{\Pi T\partial\Sigma} \mu_\partial \iota_\partial{}^*\psi \,. \tag{4.14}$$

Often, for the sake of simplicity, we shall write ψ rather than $\iota_\partial{}^*\psi$ in the right hand side. Similarly, by the boundary Stokes' theorem, if χ is a boundary de Rham superfield,

$$\oint_{\Pi T\partial\Sigma} \mu_\partial d_\partial \chi = 0 \,. \tag{4.15}$$

It is possible to define functional derivatives of functionals of de Rham superfields. Let ψ be a de Rham superfield and let $F(\psi)$ be a functional of ψ. We define the left/right functional derivative superfields $\delta_{l,r}F(\psi)/\delta\psi$ as follows. Let σ be a superfield of the same properties as ψ. Then,

$$\frac{d}{dt}F(\psi + t\sigma)\Big|_{t=0} = \int_{\Pi T\Sigma} \mu \, \sigma \frac{\delta_l F(\psi)}{\delta\psi} = \int_{\Pi T\Sigma} \mu \, \frac{\delta_r F(\psi)}{\delta\psi}\sigma \,. \tag{4.16}$$

To write $dF(\psi+t\sigma)/dt\big|_{t=0}$ as an integral on $\Pi T\Sigma$ as above, repeated applications of Stokes' theorem are required. This yields boundary terms, which must cancel out. In general, this is possible only if suitable boundary conditions are imposed on the superfields.

In the applications below, the components of the relevant de Rham superfields carry, besides the form degree, also a ghost degree. We shall limit ourselves to homogeneous superfields. A de Rham superfield ψ is said homogeneous if the sum of the form and ghost degree is the same for all its components $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ of ψ. The common value of that sum is called the (total) degree $\deg\psi$ of ψ. It is easy to see that the differential operator d and the integration operator $\int_{\Pi T\Sigma}\mu$ carry degree 1 and -2, respectively. Also, if $F(\psi)$ is a functional of a superfield ψ, then $\deg\delta_{l,r}F(\psi)/\delta\psi = \deg F - \deg\psi + 2$. Similar considerations hold for the boundary de Rham superfields. In this case, the differential operator d_∂ and the integration operator $\int_{\Pi T\partial\Sigma}\mu_\partial$ carry degree 1 and -1, respectively.

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5. The Hitchin sigma model in the presence of branes

In this section, we shall first review the formulation of the Hitchin sigma model in the absence of branes [31]. Subsequently, we shall show how to incorporate branes into it.

The Hitchin sigma model is closely related to the standard Poisson sigma model [33]–[34], of which it has the same field content. The approach used here is based on the Batalin-Vilkovisky quantization scheme [35]–[37]. To make the treatment as simple and transparent as possible, we shall use the convenient de Rham superfield formalism (cf. section 4) following the original work of Cattaneo and Felder in [35] (see also [45]–[47]). We shall limit ourselves to the lowest order in perturbation theory, since the constraints on target space geometry following from the Batalin-Vilkovisky classical master equation lead directly to Hitchin's generalized complex geometry. Quantum corrections will presumably yield a deformation of the latter, whose study is beyond the scope of this paper. We will not attempt the gauge fixing of the field theory, which, at any rate, is expected to be essentially identical to that of the ordinary Poisson sigma model as described in [35, 45]. We shall consider directly the twisted version of the Hitchin sigma model. The umtwisted version can be readily obtained by setting the H field to zero.

When branes are absent, the target space of the Hitchin sigma model is a twisted manifold (M, H) (cf. section 2). The base space is a closed oriented surface Σ.

The basic fields of the standard Hitchin sigma model are a degree 0 superembedding $x \in \Gamma(\Pi T\Sigma, M)$ and a degree 1 supersection $y \in \Gamma(\Pi T\Sigma, x^*\Pi T^*M)$, where Π is the parity inversion operator. With respect to each local coordinate t^a of M, x, y are given as de Rham superfields x^a, y_a. Under a change of coordinates, these transform as

$$x'^a = t'^a \circ t^{-1}(x) \tag{5.1}$$

$$y'_a = \partial'_a t^b \circ t^{-1}(x) y_b \,. \tag{5.2}$$

The resulting transformation rules of the de Rham components of $x^a(z, \zeta)$, $y_a(z, \zeta)$ are obtainable by expanding these relations in powers of ζ^α.

The Batalin-Vilkovisky odd symplectic form is

$$\Omega = \int_{\Pi T\Sigma} \mu \left[\delta x^a \delta y_a + \frac{1}{2} H_{abc}(x) \delta x^a dx^b \delta x^c \right]. \tag{5.3}$$

Ω is not of the canonical form when $H \neq 0$. Hence, x^a, y_a are not canonical fields/antifields. However, Ω is a closed functional form, $\delta\Omega = 0$. In this way, one can define antibrackets $(\, ,)$ in standard fashion. The resulting expression is

$$(F, G) = \int_{\Pi T\Sigma} \mu \left[\frac{\delta_r F}{\delta x^a} \frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a} \frac{\delta_l G}{\delta x^a} - H_{abc}(x) \frac{\delta_r F}{\delta y_a} dx^b \frac{\delta_l G}{\delta y_c} \right], \tag{5.4}$$

for any two functionals F, G of x^a, y_a.

In the Hitchin sigma model, the target space geometry is specified by a generalized almost complex structure $\mathcal{J}$ (cf. section 2). In the representation (2.11), the action is

$$S = \int_{\Pi T\Sigma} \mu \left[y_a dx^a + \frac{1}{2} P^{ab}(x) y_a y_b + \frac{1}{2} Q_{ab}(x) dx^a dx^b + J^a{}_b(x) y_a dx^b \right] - 2 \int_\Gamma \bar{x}^{(0)*} H \,. \tag{5.5}$$

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Here, Γ is a 3-fold such that

$$\partial\Gamma = \Sigma\,. \tag{5.6}$$

$\bar{x}^{(0)} : \Gamma \to M$ is an embedding such that $\bar{x}^{(0)}|_\Sigma$ equals the lowest degree 0 component $x^{(0)}$ of the superembedding x. The last H dependent term is a Wess–Zumino like term. A similar term was added to the action of the standard Poisson sigma model in ref. [48]. Its value depends on the the choice of $\bar{x}^{(0)}$. In the quantum theory, in order to have a well defined weight $\exp(\sqrt{-1}S)$ in the path integral, it is necessary to require that $H/2\pi$ has integer periods, so that the cohomology class $[H/2\pi] \in H^3(M,\mathbb{R})$ belongs to the image of $H^3(M,\mathbb{Z})$ in $H^3(M,\mathbb{R})$ (cf. section 2). We note that the above definition of the topological term works only if $x^{(0)}(\Sigma)$ is a boundary in M. If one wants to extend the definition to the general case where $x^{(0)}(\Sigma)$ is a cycle of M, the theory of Cheeger-Simons differential characters is required [49, 50].

A straightforward computation furnishes

$$\begin{aligned}(S,S) = 2\int_{\Pi T\Sigma}\mu\,\Big[&-\frac{1}{6}A_H{}^{abc}(x)y_ay_by_c + \frac{1}{2}B_{Ha}{}^{bc}(x)dx^ay_by_c - \\ &-\frac{1}{2}C_{Hab}{}^{c}(x)dx^adx^by_c + \frac{1}{6}D_{Habc}(x)dx^adx^bdx^c - \\ &-\frac{1}{3}H_{abc}(x)dx^adx^bdx^c\Big]\,,\end{aligned} \tag{5.7}$$

where the tensors A_H, B_H, C_H, D_H are given by (2.18a)–(2.18d). Hence, S satisfies the classical Batalin-Vilkovisky master equation

$$(S,S) = 0 \tag{5.8}$$

if (2.17a)–(2.17d) hold, i.e. when $\mathcal{J}$ is an H twisted generalized complex structure and, so, $(M,H,\mathcal{J})$ a twisted generalized complex manifold. (Recall that $dx^adx^bdx^c = 0$ on $\Pi T\Sigma$.) This shows that *there is a non trivial connection between generalized complex geometry and quantization à la Batalin–Vilkovisky of the sigma model* [31]. (2.17a)–(2.17d) are sufficient but not necessary conditions for the fulfillment of the master equation (5.8).

The Batalin–Vilkovisky variations $\delta_{BV}x^a$, $\delta_{BV}y_a$ are given by

$$\delta_{BV}x^a = (S,x^a)\,, \tag{5.9a}$$

$$\delta_{BV}y_a = (S,y_a)\,. \tag{5.9b}$$

Using (5.4), we can then derive the expressions of the Batalin-Vilkovisky variations $\delta_{BV}x^a$, $\delta_{BV}y_a$. The result is

$$\delta_{BV}x^a = dx^a + P^{ab}(x)y_b + J^a{}_b(x)dx^b\,, \tag{5.10a}$$

$$\begin{aligned}\delta_{BV}y_a = &\, dy_a + \frac{1}{2}\partial_aP^{bc}(x)y_by_c + \frac{1}{2}(\partial_aQ_{bc} + \partial_bQ_{ca} + \partial_cQ_{ab})(x)dx^bdx^c + \\ &+ (\partial_aJ^b{}_c - \partial_cJ^b{}_a)(x)y_bdx^c + J^b{}_a(x)dy_b + \\ &+ \frac{1}{2}(H_{abd}J^d{}_c - H_{acd}J^d{}_b)(x)dx^bdx^c + H_{adc}P^{db}(x)y_bdx^c\,.\end{aligned} \tag{5.10b}$$

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As well-known [36, 37], the Batalin-Vilkovisky variation operator δ_{BV} is nilpotent,

$$\delta_{BV}{}^2 = 0\,. \tag{5.11}$$

The associated cohomology is the classical Batalin-Vilkovisky cohomology. Also, by (5.8)

$$\delta_{BV} S = 0\,. \tag{5.12}$$

It is interesting to see how the odd symplectic form Ω behaves under a b transform of the H field of the form (2.13). It turns out that a meaningful comparison of the resulting symplectic form Ω' and the original symplectic form Ω requires that the superfields x^a, y_a also must undergo a b transform of the form

$$x'^a = x^a\,, \tag{5.13a}$$
$$y'_a = y_a + b_{ab}(x)dx^b\,. \tag{5.13b}$$

It is then simple to verify that

$$\Omega' = \Omega\,. \tag{5.14}$$

If the 2-form b is closed, then $H' = H$, by (2.4). In that case, the b transform is canonical, i.e. it leaves the Batalin-Vilkovisky odd symplectic form (5.3) invariant.

It is similarly interesting to see how the action S behaves under a b transform of the H field and of the generalized almost complex structure $\mathcal{J}$ of the form (2.13), (2.16a)–(2.16c). Provided the field redefinition (5.13a), (5.13b) are carried out, one finds that the resulting action S' action equals the original one S,

$$S' = S\,. \tag{5.15}$$

So, *b transform is a duality symmetry of the Hitchin sigma model* [31].[5]

The Batalin-Vilkovisky variation operator δ_{BV} behaves covariantly under b transformation in the following sense. Let H' and $\mathcal{J}'$ be a b transform of the H field and of the generalized complex structure $\mathcal{J}$ given in (2.13), (2.16a)–(2.16c). Let x'^a, y'_a be the b transform of the superfields x^a, y_a given in (5.13a, (5.13b). Let $\delta'_{BV}x'^a$, $\delta'_{BV}y'_a$ be given by (5.10a), (5.13b) with x^a, y_a and H_{abc}, P^{ab}, $J^a{}_b$ Q_{ab} replaced by x'^a, y'_a H'_{abc}, P'^{ab}, $J'^a{}_b$ Q'_{ab}, respectively. Then, one has the relations

$$\delta'_{BV}x'^a = \delta_{BV}x^a\,, \tag{5.16a}$$
$$\delta'_{BV}y'_a = \delta_{BV}\Big(y_a + b_{ab}(x)dx^b\Big)\,. \tag{5.16b}$$

(Compare (5.16a), (5.16b) with (5.13a), (5.13b).) In this sense, *b transformation is compatible with the Batalin-Vilkovisky δ_{BV} cohomological structure.*

[5] In the untwisted Hitchin model, the H field is absent and the action S has no topological term. In that case the 2-form b must be closed and $S' = S - 2\int_\Sigma x^{(0)*}b$, i.e. S, S' differ by a topological term. If $b/2\pi$ has integer periods and, so, describes a gerbe gauge transformation, one has $\exp(\sqrt{-1}S') = \exp(\sqrt{-1}S)$ in the quantum path integral. b transform is then a duality symmetry of the quantum Hitchin sigma model. See [31].

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Let see next how one can incorporate branes into the Hitchin sigma model. The discussion at the end of section 3 shows that, if we want to make contact with the usual picture of branes, a brane contained in the target bulk space should not be seen simply as an ordinary submanifold W of the manifold M, but rather as a generalized submanifold (W, F) of the twisted manifold (M, H).

A sigma model with branes describes the dynamics of open strings whose ends lie on the branes. Therefore, the world sheet Σ of our sigma model with branes must have a non empty boundary $\partial\Sigma$ and the superfields x, y must satisfy appropriate boundary conditions. These are properly expressed in terms of the pull-back superfields $\iota_\partial{}^* x$, $\iota_\partial{}^* y$ (cf. section 4). Below, for notational simplicity, $\iota_\partial{}^*$ will often be tacitly understood.

The superembedding x must be such that $\iota_\partial{}^* x(\Pi T\partial\Sigma) \subseteq W$. As a consequence, the superfield x^a must satisfy the boundary condition

$$x^\rho = 0 \tag{5.17a}$$

along $\Pi T\partial\Sigma$, in adapted coordinates (cf. section 3). From a geometrical point of view, it is also natural to demand that the pull–back superfields $\iota_\partial{}^* dx$, $\iota_\partial{}^* y$ represent a section of the pull-back $(\iota_\partial{}^* x)^* \Pi \mathcal{T}^F W$ of the parity reversed generalized tangent bundle $\Pi \mathcal{T}^F W$ of W (cf. section 3). On account of (3.6a), (3.6b), one has the further boundary condition

$$y_i + F_{ij}(x) d_\partial x^j = 0 \tag{5.17b}$$

along $\Pi T\partial\Sigma$, in adapted coordinates. (The condition $d_\partial x^\rho = 0$ is already implied by (5.17a).) The boundary conditions (5.17a), (5.17b) are purely geometrical hence *kinematical*.

When M contains a brane W, the expression (5.3) of the Batalin-Vilkovisky odd symplectic Ω is no longer correct. As it stands, Ω fails to be closed, as in fact

$$\delta\Omega = -\oint_{\Pi T\partial\Sigma} \mu_\partial \, \frac{1}{2} \partial_i F_{jk}(x) \delta x^i \delta x^j \delta x^k \,. \tag{5.18}$$

The right hand side is a boundary term and can be compensated for by modifying (5.3) by a boundary term. The resulting expression is

$$\Omega_W = \int_{\Pi T\Sigma} \mu \left[\delta x^a \delta y_a + \frac{1}{2} H_{abc}(x) \delta x^a dx^b \delta x^c \right] + \oint_{\Pi T\partial\Sigma} \mu_\partial \, \frac{1}{2} F_{ij}(x) \delta x^i \delta x^j \,. \tag{5.19}$$

Now, Ω_W is a closed functional form, $\delta\Omega_W = 0$. In this way, one can define antibrackets $(\,,)_W$ in standard fashion. The resulting expression is

$$\begin{aligned}(F, G)_W = & \int_{\Pi T\Sigma} \mu \left[\frac{\delta_r F}{\delta x^a} \frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a} \frac{\delta_l G}{\delta x^a} - H_{abc}(x) \frac{\delta_r F}{\delta y_a} dx^b \frac{\delta_l G}{\delta y_c} \right] - \\ & - \oint_{\Pi T\partial\Sigma} \mu_\partial \, F_{ij}(x) \frac{\delta_r F}{\delta y_i} \frac{\delta_l G}{\delta y_j} \,, \end{aligned} \tag{5.20}$$

for any two functionals F, G of x^a, y_a.

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Now, let us introduce the Hitchin sigma model in the presence of a brane (W, F). The target space geometry of the bulk (M, H) is specified again by a generalized almost complex structure $\mathcal{J}$. (W, F) is assumed to be a generalized almost complex submanifold of $(M, H, \mathcal{J})$ (cf. section 3). This assumption is natural from a geometrical point of view. In the representation (2.11), the action of the model reads

$$S_W = \int_{\Pi T\Sigma} \mu \left[y_a dx^a + \tfrac{1}{2} P^{ab}(x) y_a y_b + \frac{1}{2} Q_{ab}(x) dx^a dx^b + J^a{}_b(x) y_a dx^b \right] - \\ -2 \int_\Gamma \bar{x}^{(0)*} H + 2 \int_\Delta \bar{x}^{(0)*} F \,. \tag{5.21}$$

Here, Γ, Δ are respectively a 3– and a 2–fold such that $\Delta \subseteq \partial\Gamma$,

$$\partial\Gamma - \Delta = \Sigma \,, \qquad \partial\Delta = -\partial\Sigma \tag{5.22}$$

$\bar{x}^{(0)} : \Gamma \to M$ is an embedding such that $\bar{x}^{(0)}(\Delta) \subseteq W$ and that $\bar{x}^{(0)}|_\Sigma$ equals the lowest degree 0 component $x^{(0)}$ of the superembedding x. The last two terms dependent on H, F constitute altogether a Wess–Zumino like term. Their total value depends on the choice of $\bar{x}^{(0)}$. In the quantum theory, in order to have a well defined weight $\exp(\sqrt{-1} S_W)$ in the path integral, it is necessary to require that $(H/2\pi, F/2\pi)$ has integer relative periods, so that the relative cohomology class $[(H/2\pi, F/2\pi)]$ belongs to the image of $H^3(M, W, \mathbb{Z})$ in $H^3(M, W, \mathbb{R})$ (cf. section 3). The above definition of the topological term works only if $(x^{(0)}(\Sigma), x^{(0)}(\partial\Sigma))$ is a relative boundary in M. If one wants to extend the definition to the general case where $(x^{(0)}(\Sigma), x^{(0)}(\partial\Sigma))$ is a relative cycle of M, the theory of Cheeger-Simons relative differential characters is required [51].

The brane action S_W given in (5.21) is the most obvious generalization of the no brane action S given in (5.5). It is so designed to yield the same expression for the derivatives $\frac{\delta_{l,r} S_W}{\delta x^a}$, $\frac{\delta_{l,r} S_W}{\delta y_a}$ of S_W and the derivatives $\frac{\delta_{l,r} S}{\delta x^a}$, $\frac{\delta_{l,r} S}{\delta y_a}$ of S upon imposing a natural boundary condition on the fields as we show next. Using the general definition (4.16) to compute $\frac{\delta_{l,r} S_W}{\delta x^a}$, one obtains an integral on $\Pi T\Sigma$ and a boundary integral on $\Pi T\partial\Sigma$ of the form

$$B.I. = -\oint_{\Pi T\partial\Sigma} \mu_\partial \, F_{ij}(x) \left[d_\partial x^j + J^j{}_k(x) d_\partial x^k + P^{j\rho}(x) y_\rho - P^{jk} F_{kl}(x) d_\partial x^l \right] \delta x^i \,, \tag{5.23}$$

in adapted coordinates. The calculation involves repeated applications of the relations (3.7a)–(3.7c) which follow from (W, F) being a generalized almost complex submanifold of $(M, H, \mathcal{J})$ and the kinematical boundary conditions (5.17a), (5.17b). From (5.23), it seems reasonable to demand that

$$F_{ij}(x) \left[d_\partial x^j + J^j{}_k(x) d_\partial x^k + P^{j\rho}(x) y_\rho - P^{jk} F_{kl}(x) d_\partial x^l \right] = 0 \tag{5.24}$$

along $\Pi T\partial\Sigma$.[6] Unfortunately, this boundary condition suffers a number of diseases to be discussed below. These are cured by replacing (5.24) by a stronger boundary condition implying (5.24), namely

$$d_\partial x^i + J^i{}_j(x) d_\partial x^j + P^{i\rho}(x) y_\rho - P^{ij} F_{jk}(x) d_\partial x^k = 0 \tag{5.25}$$

[6] Of course, this condition is no longer necessary when $F_{ij} = 0$. We shall not consider this possibility in the following discussion.

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along $\Pi T\partial\Sigma$.[7] Using again the general definition (4.16) to compute $\frac{\delta_{l,r}S_W}{\delta y_a}$, one obtains again an integral on $\Pi T\Sigma$ and no boundary integral on $\Pi T\partial\Sigma$. Therefore, no further boundary conditions are required. The boundary conditions (5.24), (5.25) originate from the structure of the brane action S_W and are therefore *dynamical*.

The expressions of $\frac{\delta_{l,r}S_W}{\delta x^a}$, $\frac{\delta_{l,r}S_W}{\delta y_a}$ obtained in this way are given respectively by the right hand side of (5.10b) and plus/minus the the right hand side of (5.10a). Using these, one can compute the antibrackets $(S_W, S_W)_W$ using (5.20). The calculation involves repeated applications of the relations (3.7a)–(3.7c) and the kinematical boundary conditions (5.17a), (5.17b) again, and the dynamical boundary condition (5.24) or (5.25). (One also uses that $d_\partial x^i d_\partial x^j = 0$ on the 1–dimensional manifold $\partial\Sigma$.) Because of (5.24), the boundary term in the right hand side of (5.20) vanishes identically. The result is

$$
\begin{aligned}
(S_W, S_W)_W = 2\int_{\Pi T\Sigma} \mu \Big[-\frac{1}{6} A_H{}^{abc}(x) y_a y_b y_c + \frac{1}{2} B_{Ha}{}^{bc}(x) dx^a y_b y_c - \\
- \frac{1}{2} C_{Hab}{}^{c}(x) dx^a dx^b y_c + \frac{1}{6} D_{Habc}(x) dx^a dx^b dx^c - \\
- \frac{1}{3} H_{abc}(x) dx^a dx^b dx^c \Big] ,
\end{aligned}
\tag{5.26}
$$

where the tensors A_H, B_H, C_H, D_H are given by (2.18a)–(2.18d). Note that the expression (5.26) of the brackets $(S_W, S_W)_W$ is formally identical to the expression (5.7) of the brackets (S, S). Hence, S_W satisfies the brane classical Batalin-Vilkovisky master equation

$$(S_W, S_W)_W = 0\,, \tag{5.27}$$

if (2.17a)–(2.17d) hold, i.e. when $\mathcal{J}$ is an H twisted generalized complex structure. In this way, the brane (W, F) is a generalized complex submanifold of the twisted generalized complex manifold $(M, H, \mathcal{J})$ (cf. section 3). This shows that *the non trivial connection between generalized complex geometry and quantization à la Batalin-Vilkovisky of the sigma model continues to hold also when M contains a brane W.* This is so by design.

The Batalin-Vilkovisky variations $\delta_{BVW}x^a$, $\delta_{BVW}y_a$ are given by

$$\delta_{BVW}x^a = (S_W, x^a)_W\,, \tag{5.28a}$$

$$\delta_{BVW}y_a = (S_W, y_a)_W\,. \tag{5.28b}$$

Using (5.20), we can then derive the expressions of the brane Batalin-Vilkovisky variations $\delta_{BVW}x^a$, $\delta_{BVW}y_a$. They are given by the same formulae as those of the no brane Batalin-Vilkovisky variations $\delta_{BV}x^a$, $\delta_{BV}y_a$ given in (5.10a), (5.10b). Again, this is so by design. As δ_{BV} (cf. eq. (5.11)), the brane Batalin-Vilkovisky variation operator δ_{BVW} is nilpotent

$$\delta_{BVW}{}^2 = 0\,. \tag{5.29}$$

The associated cohomology is the brane classical Batalin-Vilkovisky cohomology. This relation is not the trivial restatement of (5.11), which it may appear at first glance, since,

[7] Exploiting (5.17a), (5.17b), it is straightforward to show that both conditions (5.24), (5.25) are adapted covariant.

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in the presence of branes, the superfields x^a, y_a obey the boundary conditions derived above. (The natural question about the compatibility of the boundary conditions with the Batalin-Vilkovisky cohomological structure will be discussed in the next section.) Also, one has

$$\delta_{BVW} S_W = 0\,, \tag{5.30}$$

by (5.27).

It is interesting to check whether the incorporation of branes into the Hitchin sigma model is compatible with the b transformation symmetry.

To begin with, let us see how the odd symplectic form Ω_W behaves under a b transform of the H and F fields of the form (2.13), (3.5). Again, a meaningful comparison of the resulting symplectic form Ω'_W and the original symplectic form Ω_W requires that the superfields x^a, y_a undergo the b transform (5.13a), (5.13b). From (5.19), it is straightforward to verify that

$$\Omega'_W = \Omega_W\,, \tag{5.31}$$

generalizing (5.14).

Next, let us see how the action S_W behaves under a b transform of the H and F fields and of the generalized almost complex structure $\mathcal{J}$ of the form (2.13), (3.5), (2.16a)–(2.16c). Using (5.13a), (5.13b), one finds that

$$S'_W = S_W\,, \tag{5.32}$$

generalizing (5.15). So, *b transform is a duality symmetry of the Hitchin sigma model even in the presence of a brane.*

Expectedly, the brane Batalin-Vilkovisky variation operator δ_{BVW} behaves covariantly under b transformation as in the no brane case. Relations (5.16a), (5.16b) hold true with δ_{BV}, δ'_{BV} replaced by δ_{BVW}, δ'_{BVW}, respectively, where δ'_{BVW} is defined analogously to δ'_{BV}. Thus, *b transformation is compatible with the Batalin-Vilkovisky cohomological structure also in the brane case.*

In the presence of a brane W, the superfields x^a, y_a obey the boundary conditions (5.17a), (5.17b) and (5.24) or (5.25). It is important to check whether these conditions are invariant under b transformation.

Let us consider first the kinematical boundary conditions (5.17a), (5.17b). Requiring their b invariance amounts to demanding that whenever the x^a, y_a satisfy (5.17a), (5.17b) with respect to the brane field F, their b transforms x'^a, y'_a, given by (5.13a), (5.13b), satisfy (5.17a), (5.17b) with respect to the b transformed brane field F', given by (3.5). It is trivial to check that this is indeed the case.

Let us consider next the dynamical boundary conditions (5.24) or (5.25). Again, requiring their b invariance amounts to demanding that whenever the x^a, y_a satisfy (5.24) or (5.25) with respect to the fields H and F and the generalized almost complex structure $\mathcal{J}$, their b transforms x'^a, y'_a, given by (5.13a), (5.13b), satisfy (5.24) or (5.25) with respect to the b transformed fields H' and F' and the b transformed generalized almost complex structure $\mathcal{J}'$, given by (2.13), (3.5), (2.16a)–(2.16c). It is trivial to check that this is the case for the boundary condition (5.25) but not for the boundary condition (5.24). The

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latter therefore breaks b invariance. b transformation symmetry is one of the most basic features of both generalized complex geometry and the Hitchin sigma model. Renouncing to it seems to be out of question. This indicates that *(5.25) is the appropriate dynamical boundary condition.*

6. The classical BV cohomology in the presence of branes

One of the most interesting aspects of the Hitchin sigma model is the associated classical Batalin-Vilkovisky cohomology, which describes the observables of the model at semiclassical level [36, 37]. As we have seen above, when a brane W is present, the expressions of the brane Batalin-Vilkovisky variations $\delta_{BVW}x^a$, $\delta_{BVW}y_a$ are the same as those of the no brane variations $\delta_{BV}x^a$, $\delta_{BV}y_a$ given in (5.10a), (5.10b). One would thus conclude that the no brane and brane classical Batalin-Vilkovisky cohomology are identical. However, one should keep in mind that, in the brane case the basis fields x^a, y_a of the model obey the kinematical boundary conditions (5.17a), (5.17b) and the dynamical boundary conditions (5.24) or (5.25). It is therefore important to check the compatibility of the boundary conditions and the overall Batalin-Vilkovisky cohomological structure.

Concretely, this is done as follows. The boundary conditions have the general form

$$\mathcal{F}(\iota_\partial{}^*x, \iota_\partial{}^*y) = 0\,, \tag{6.1}$$

where $\mathcal{F}$ is some functional of the boundary superfields. The boundary conditions are compatible with Batalin-Vilkovisky cohomological structure provided

$$\delta_{BVW}\mathcal{F}(\iota_\partial{}^*x, \iota_\partial{}^*y) = 0\,, \tag{6.2}$$

when (6.1) holds, where the left hand side is computed using (5.10a), (5.10b) and the fact that $\delta_{BVW}\iota_\partial{}^* = \iota_\partial{}^*\delta_{BVW}$. If (6.2) does not hold, then (6.2) becomes a new set of boundary conditions to be added to the original ones. The compatibility check then must be carried out again from the beginning with the enlarged set of boundary conditions so obtained. The process must be continued until (6.2) is satisfied identically. Alternatively, one may replace the boundary conditions (6.1) with stronger boundary conditions implying (6.1), for which (6.2) holds identically. (These will be expressed in terms of some functional $\hat{\mathcal{F}}$ different from $\mathcal{F}$.)

We consider first the kinematical boundary conditions (5.17a), (5.17b). Their compatibility with the Batalin-Vilkovisky cohomological structure requires that

$$\delta_{BVW}x^\rho = 0\,, \tag{6.3a}$$

$$\delta_{BVW}y_i - F_{ij}(x)d_\partial\delta_{BVW}x^j + \partial_kF_{ij}(x)\delta_{BVW}x^k d_\partial x^j = 0 \tag{6.3b}$$

along $\Pi T\partial\Sigma$, in adapted coordinates, when (5.17a), (5.17b) hold. Using (3.7a)–(3.7c) and (5.17a), (5.17b) it is straightforward to verify that these relations do indeed hold true.

We consider next the dynamical boundary conditions (5.24) or (5.25). In the previous section, we saw that the basic requirement of b transformation symmetry invariance of the dynamical boundary conditions rules out (5.24) and selects (5.25) as the only consistent

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condition. We now shall show that the requirement of compatibility of the dynamical boundary conditions with the Batalin–Vilkovisky cohomological structure leads to the very same conclusion.

By (5.10a), the condition (5.24) can be written concisely as

$$F_{ij}(x)\delta_{BVW}x^j = 0 \tag{6.4}$$

along $\Pi T\partial\Sigma$. Proceeding as above, its compatibility with the Batalin-Vilkovisky cohomological structure requires that

$$\partial_k F_{ij}(x)\delta_{BVW}x^k\delta_{BVW}x^j = 0 \tag{6.5}$$

along $\Pi T\partial\Sigma$, when (5.17a), (5.17b) and (5.24) hold. Explicitly written, (6.5) reads

$$(\partial_k F_{ij} + \partial_j F_{ki})\, P^{k\sigma}(x)y_\sigma \left[d_\partial x^j + J^j{}_l(x)d_\partial x^l + \frac{1}{2}P^{j\rho}(x)y_\rho - P^{jl}F_{lm}(x)d_\partial x^m\right] = 0 \tag{6.6}$$

along $\Pi T\partial\Sigma$. (6.6) is not satisfied identically, when (5.17a), (5.17b) and (5.24) hold. According to the procedure described above, (6.6) should be added as a new boundary condition. From (5.29), (6.5), it is easy to see that no further boundary conditions would be required by compatibility. The alternative condition (5.25) can be written concisely

$$\delta_{BVW}x^j = 0 \tag{6.7}$$

along $\Pi T\partial\Sigma$. By (5.29), (6.7), the boundary condition (5.25) satisfies the compatibility requirements (6.2) trivially.

In conclusion, (5.24) involves the introduction of the new dynamical boundary condition (6.6). (6.6) is rather complicated and does not seem to have any obvious natural interpretation. (5.25), conversely, does not have this undesirable feature. This analysis provides conclusive evidence that (5.25), not (5.24), is the appropriate consistent dynamical boundary condition of Hitchin sigma model. Combining (6.3a), (6.7) one gets the boundary condition

$$\delta_{BVW}x^a = 0 \tag{6.8}$$

along $\Pi T\partial\Sigma$, where the left hand side is given by (5.10a). This seems to be a more transparent set of boundary conditions, also in the light of the fact that the y classical field equations of the Hitchin sigma model can be written precisely as $\delta_{BVW}x^a = 0$ on $\Pi T\Sigma$.

In ref. [31], the classical Batalin-Vilkovisky cohomology of the no brane Hitchin sigma model was studied. In particular, it was shown that there exists a natural homomorphism of the generalized Dolbeault cohomology of target space into the Batalin-Vilkovisky cohomology depending on a choice of a singular supercycle Z of the world sheet Σ. The analysis was based on an adaptation of the well-known descent formalism of gauge theory. The homomorphism was constructed by associating with a representative of a generalized Dolbeault cohomology class a local superfield and integrating the latter on Z and by showing that the results was a representative of a Batalin-Vilkovisky cohomology class. It was also found that the generalized Dolbeault cohomology contains the cohomology of the

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generalized deformation complex as a subcohomology. We shall not repeat the details of this analysis. The interested reader is invited to read of ref. [31, section 6] for a thorough discussion of these matters.

One may wonder whether and how the presence of branes modifies the classical Batalin-Vilkovisky cohomology. One may think, for instance, that a relative version of the generalized Dolbeault cohomology may be the appropriate cohomology in this case, since relative cohomology involves typically a space and a subspace, here the target bulk and the brane. This is unlikely to work out for the following simple reason. A generalized complex submanifold of a twisted generalized complex manifold is *not* itself a twisted generalized complex manifold in general, while any reasonable relative version of a cohomology requires that the two spaces involved carry the same type of structure.

A more likely scenario is the following. In the presence of branes, the Batalin-Vilkovisky cohomology will contain a sector reproducing the no brane cohomology associated with the generalized Dolbeault cohomology of the bulk and a further sector based on some cohomological structure of the brane, whose nature is to be determined. We devote the rest of this section to the analysis of this matter. The discussion parallels that of the no brane cohomology of ref. [31] with some significant differences to be discussed below.

The construction expounded in the following involves the singular chain complex of $\partial\Sigma$. Since we deal with boundary de Rham superfields throughout (cf. section 4), it is convenient to use the boundary singular superchains formalism. A boundary singular superchain C is a doublet formed by a 0-, 1-dimensional singular chain $C_{(0)}$, $C_{(1)}$ of $\partial\Sigma$ organized as a formal chain sum $C = C_{(0)} + C_{(1)}$. The nilpotent singular boundary operator ∂ of $\partial\Sigma$ extends to boundary superchains in obvious fashion by setting $(\partial_\partial C)_{(0)} = \partial C_{(1)}$, $(\partial_\partial C)_{(1)} = 0$. A boundary singular supercycle Z is a superchain such that $\partial_\partial Z = 0$. A boundary de Rham superfield χ can be integrated on a boundary superchain C:

$$\int_C \chi = \int_{C_{(0)}} \chi^{(0)} + \int_{C_{(1)}} ds \chi^{(1)}{}_s(s)\,. \tag{6.9}$$

Stokes' theorem holds, $\int_C d_\partial \chi = \int_{\partial_\partial C} \chi$. In particular, one has $\int_Z d_\partial \chi = 0$ for any boundary supercycle Z.

We call a boundary de Rham superfield X local, if it is a local functional of the pull–back of the basic superfields $\iota_\partial{}^* x$, $\iota_\partial{}^* y$. Let X be some local boundary superfield and let there be another local boundary superfield Y such that

$$\delta_{BVW} X = d_\partial Y\,. \tag{6.10}$$

Thus, X defines a mod d_∂ Batalin-Vilkovisky cohomology class. Then, if Z is a boundary singular supercycle, one has

$$\delta_{BVW} \int_Z X = \int_Z d_\partial Y = 0\,, \tag{6.11}$$

by Stokes' theorem. It follows that

$$\langle Z, X \rangle_\partial = \int_Z X \tag{6.12}$$

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defines a Batalin-Vilkovisky cohomology class. A standard analysis shows that this class depends only on the mod d_∂ Batalin-Vilkovisky cohomology class of X. So, one may obtain Batalin-Vilkovisky cohomology classes by constructing boundary local superfields X satisfying (6.10).

Define

$$\partial_\partial = \frac{1}{2}\left[d_\partial - \sqrt{-1}(\delta_{BVW} - d_\partial)\right] \tag{6.13}$$

and its complex conjugate $\overline{\partial}_\partial$, where here and below the operator δ_{BVW} is tacitly restricted to the space of local boundary superfields. From (5.29), using that $d_\partial\delta_{BVW} + \delta_{BVW}d_\partial = 0$, it is immediate to check that

$$\partial_\partial{}^2 = 0\,, \tag{6.14a}$$

$$\overline{\partial}_\partial{}^2 = 0\,, \tag{6.14b}$$

$$\partial_\partial\overline{\partial}_\partial + \overline{\partial}_\partial\partial_\partial = 0\,. \tag{6.14c}$$

From (6.5), one has further

$$d_\partial = \partial_\partial + \overline{\partial}_\partial\,, \tag{6.15a}$$

$$\delta_{BVW} = \partial_\partial + \overline{\partial}_\partial + \sqrt{-1}(\partial_\partial - \overline{\partial}_\partial)\,. \tag{6.15b}$$

The operator $\overline{\partial}_\partial$ acts on the space of local boundary superfields, carries degree 1, by (6.13), and it squares to 0, by (6.14b). Therefore, one can define a $\overline{\partial}_\partial$ local boundary superfield cohomology in obvious fashion.

Let X be a local boundary superfield such that

$$\overline{\partial}_\partial X = 0\,. \tag{6.16}$$

X defines a $\overline{\partial}_\partial$ local boundary superfield cohomology class. By (6.15a), (6.15b), X satisfies (6.10) with $Y = (1 + \sqrt{-1})X$. So, as shown above, for any boundary supercycle Z, $\langle Z, X\rangle_\partial$ defines a Batalin-Vilkovisky cohomology class. If $X = \overline{\partial}_\partial U$ for some local boundary superfield U, so that the corresponding $\overline{\partial}_\partial$ cohomology class is trivial, then $\langle Z, X\rangle_\partial = \frac{1}{2}\sqrt{-1}\delta_{BVW}\langle Z, U\rangle_\partial$, by (6.13) and Stokes' theorem and, so, the corresponding Batalin-Vilkovisky class is trivial too. Therefore, for any boundary singular supercycle Z, there is a well-defined homomorphism from the $\overline{\partial}_\partial$ local boundary superfield cohomology into the Batalin-Vilkovisky cohomology. This homomorphism depends only on the boundary singular homology class of Z.

Adapting the analysis of ref. [31], we carry out the construction of the brane sector of the Batalin-Vilkovisky cohomology by associating with a given field of the brane a local boundary superfield and imposing that the latter satisfies (6.16). Taking into account the boundary conditions (5.17a), (5.17b), we see that the basic boundary superfields at our disposal are the x^i, y_ρ, in adapted coordinates. (Here and below, the pull-back operator $\iota_\partial{}^*$ is tacitly understood.) Tentatively, we may consider local boundary superfields of the form

$$X_\Upsilon = \sum_{p,q\geq 0} \frac{1}{p!q!}\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q}(x)y_{\rho_1}\ldots y_{\rho_p}d_\partial x^{i_1}\ldots d_\partial x^{i_q}\,, \tag{6.17}$$

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where $\Upsilon \in \oplus_{p,q\geq 0} C^\infty(\wedge^p NW \otimes \wedge^q T^*W \otimes \mathbb{C})$, $NW = TM|_W/TW$ being the normal bundle of W. This is however too naive. In fact, under a change of adapted coordinates (cf. section 3), the $d_\partial x^i$ transform into themselves, since $d_\partial x^\rho = 0$ by (5.17a), while the y_ρ do not but mix with the $F_{ij}(x)d_\partial x^j$, since $y_i = -F_{ij}(x)d_\partial x^j$ by (5.17b). For this reason, an object of the form (6.17) would not be adapted covariant (cf. section 3). To get adapted covariant expressions, the components $\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q}$ of the brane field Υ must have more exotic transformation properties than those simplemindedly implied above. Formally, these are given as follows. $\Upsilon \in \mathcal{X}_{W0}{}^* = \oplus_{n\geq 0}\mathcal{X}_{W0}{}^n$, where $\mathcal{X}_{W0}{}^n$ is the space of the formal adapted covariant objects

$$\Upsilon = \sum_{p,q\geq 0, p+q=n} \frac{1}{p!q!} \Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} \eta_{\rho_1}\ldots\eta_{\rho_p}\xi^{i_1}\ldots\xi^{i_q} \tag{6.18}$$

ξ^i, η_ρ being anticommuting objects transforming as

$$\xi'^i = \partial_j t'^i \xi^j\,, \tag{6.19a}$$

$$\eta'_\rho = \partial'_\rho t^\sigma \eta_\sigma - \partial'_\rho t^i F_{ij}\xi^j\,, \tag{6.19b}$$

under a change of adapted coordinates. The resulting transformation properties of the component fields $\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q}$ are *not* tensorial and in explicit form are rather messy.

We note that all terms in the right hand side of (6.17) with $q > 1$ vanish, since $d_\partial x^i d_\partial x^j = 0$ on the 1-dimensional manifold $\partial\Sigma$. This property will be used repeatedly to suitably reshape the expressions derived below.

A simple calculation using (5.10a), (5.10b), (3.7a)–(3.7c), (5.17a), (5.17b) yields

$$\delta_{BVW} x^i = d_\partial x^i + P^{i\rho}(x)y_\rho + \left(J^i{}_j - P^{ik}F_{kj}\right)(x)d_\partial x^j\,, \tag{6.20a}$$

$$\begin{aligned}\delta_{BVW} y_\rho = {} & d_\partial y_\rho + \frac{1}{2}\partial_\rho P^{\sigma\tau}(x)y_\sigma y_\tau + \big(\partial_\rho J^\sigma{}_i - \partial_i J^\sigma{}_\rho + P^{\sigma j}H_{ij\rho} - F_{ij}\partial_\rho P^{j\sigma}\big)(x)\times \\ & \times y_\sigma dx^i + J^\sigma{}_\rho(x)d_\partial y_\sigma\,. \end{aligned}\tag{6.20b}$$

From these relations and (6.13), (6.17), one finds

$$\begin{aligned}\overline{\partial}_\partial X_\Upsilon = {} & \sum_{p,q\geq 0}\frac{1}{p!q!}\overline{\partial}_W \Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q}(x) y_{\rho_1}\cdots y_{\rho_p} d_\partial x^{i_1}\cdots d_\partial x^{i_q} + \\ & + \sum_{p,q\geq 0}\frac{1}{p!q!} K_W{}^\sigma \Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q}(x) d_\partial y_\sigma y_{\rho_1}\cdots y_{\rho_p} d_\partial x^{i_1}\cdots d_\partial x^{i_q}\,, \end{aligned}\tag{6.21}$$

where

$$\begin{aligned}&\overline{\partial}_W \Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} = \\ &= \frac{1}{2}\bigg\{(-1)^p q\Big[\partial_{[i_1}\Upsilon^{\rho_1\ldots\rho_p}{}_{i_2\ldots i_q]} + \sqrt{-1}\left(J^j{}_{[i_1} - P^{jk}F_{k[i_1}\right)\partial_{|j|}\Upsilon^{\rho_1\ldots\rho_p}{}_{i_2\ldots i_q]}\Big] + \\ &\quad + \frac{1}{2}(-1)^p q(q-1)\sqrt{-1}M_{[i_1}{}^j{}_{i_2}\Upsilon^{\rho_1\ldots\rho_p}{}_{|j|i_3\ldots i_q]} + \\ &\quad + (-1)^p pq\sqrt{-1}M_{[i_1}{}^{[\rho_1}{}_{|\sigma|}\Upsilon^{|\sigma|\rho_2\ldots\rho_p]}{}_{i_2\ldots i_q]} - \end{aligned}$$

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$$
\begin{aligned}
&- p\sqrt{-1}\Big[P^{[\rho_1|j|}\partial_j\Upsilon^{\rho_2\ldots\rho_p]}{}_{i_1\ldots i_q} - \frac{1}{2}(p-1)\partial_\sigma P^{[\rho_1\rho_2}\Upsilon^{|\sigma|\rho_3\ldots\rho_p]}{}_{i_1\ldots i_q} - \\
&\qquad - q\partial_{[i_1}P^{j[\rho_1}\Upsilon^{\rho_2\ldots\rho_p]}{}_{|j|i_2\ldots i_q]}\Big] + \\
&+ \frac{1}{2}q(q-1)\sqrt{-1}Z_{[i_1 i_2|\sigma|}\Upsilon^{\sigma\rho_1\ldots\rho_p}{}_{i_3\ldots i_q]}\Big\}\,, \qquad (6.22a)
\end{aligned}
$$

with

$$
\begin{aligned}
M_i{}^k{}_j &= -\partial_i\left(J^k{}_j - P^{kl}F_{lj}\right) + \partial_j\left(J^k{}_i - P^{kl}F_{li}\right), &(6.22b)\\
M_i{}^\sigma{}_\rho &= \partial_i J^\sigma{}_\rho - \partial_\rho J^\sigma{}_i - P^{\sigma j}H_{ij\rho} + F_{ij}\partial_\rho P^{j\sigma}\,, &(6.22c)\\
Z_{ij\rho} &= \partial_i Q_{j\rho} + \partial_j Q_{\rho i} + \partial_\rho Q_{ij} + \partial_i\left(F_{jk}J^k{}_\rho\right) - \partial_j\left(F_{ik}J^k{}_\rho\right) + \\
&\quad + F_{ik}\partial_\rho J^k{}_j - F_{jk}\partial_\rho J^k{}_i + F_{ik}F_{jl}\partial_\rho P^{kl} - \\
&\quad - \left(J^k{}_i - P^{kl}F_{li}\right)H_{jk\rho} + \left(J^k{}_j - P^{kl}F_{lj}\right)H_{ik\rho}\,, &(6.22d)
\end{aligned}
$$

the brackets $[\cdots]$ denoting full antisymmetrization of all enclosed indices except for those between bars $|\cdots|$, and

$$
\begin{aligned}
K_W{}^\sigma\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} = \frac{1}{2}\Big\{&\left(\delta^\sigma{}_\tau + \sqrt{-1}J^\sigma{}_\tau\right)\Upsilon^{\tau\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} + \\
&+ (-1)^p\sqrt{-1}P^{\sigma j}\Upsilon^{\rho_1\ldots\rho_p}{}_{j i_1\ldots i_q}\Big\}. \qquad (6.22e)
\end{aligned}
$$

So, X_Υ satisfies (6.16), if

$$
\begin{aligned}
\overline{\partial}_W\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} &= 0\,, &(6.23a)\\
K_W{}^\sigma\Upsilon^{\rho_1\ldots\rho_p}{}_{i_1\ldots i_q} &= 0\,. &(6.23b)
\end{aligned}
$$

It can be verified that eqs. (6.23a), (6.23b) are jointly adapted covariant. Note that they are so only together, since in fact eq. (6.23a) is not by itself adapted covariant if eq. (6.23b) is not taken into account.

It is straightforward to show that, for fixed ρ, eq. (6.22e) defines a linear operator $K_W{}^\rho : \mathcal{X}_{W0}{}^n \to \mathcal{X}_{W0}{}^{n-1}$, for $n \geq 0$. Let $\mathcal{X}_W{}^n$ be the intersection of the kernels of the operators $K_W{}^\rho$ with $\rho = 1, \ldots, d - \dim W$, for $n \geq 0$. Let further $\mathcal{X}_W{}^* = \oplus_{n\geq 0}\mathcal{X}_W{}^n$. Eq. (6.23b) then states that $\Upsilon \in \mathcal{X}_W{}^*$.

Using (6.22a)–(6.22e), by a very lengthy algebraic verification exploiting systematically (2.17a)–(2.17d) and (3.7a)–(3.7c), one finds that eqs. (6.22a)–(6.22d) define a linear operator $\overline{\partial}_W : \mathcal{X}_W{}^n \to \mathcal{X}_W{}^{n+1}$ and that

$$
\overline{\partial}_W{}^2 = 0 \qquad \text{on } \mathcal{X}_W{}^*\,. \qquad (6.24)
$$

Thus, the pair $(\mathcal{X}_W{}^*, \overline{\partial}_W)$ is a cochain complex, with which there is associated a cohomology $H^*(\mathcal{X}_W{}^*, \overline{\partial}_W)$, which we call the generalized complex submanifold cohomology of W. Eq. (6.23a) then states that $\Upsilon \in \mathcal{X}_W{}^*$ is a representative of a class of this cohomology.

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It is easy to see that *eq. (6.17) defines a homomorphism of the generalized complex submanifold cohomology of the brane W into the $\overline{\partial}_\partial$ local boundary superfield cohomology.* Recall that the latter is embedded in a sector of the classical Batalin-Vilkovisky cohomology associated with the presence of branes. Therefore, *the brane sector of the classical Batalin-Vilkovisky cohomology is related non trivially to the generalized complex submanifold cohomology of W.*

$\mathcal{X}_W{}^*$ is actually a graded algebra and $\overline{\partial}_W$ is a derivation on this algebra. As a consequence, $H^*(\mathcal{X}_W{}^*, \overline{\partial}_W)$ has a canonical ring structure. The cohomology ring $H^*(\mathcal{X}_W{}^*, \overline{\partial}_W)$ characterizes the generalized complex submanifold W. To the best of our knowledge, it was hitherto unknown.

The investigation of the intrinsic content of $H^*(\mathcal{X}_W{}^*, \overline{\partial}_W)$, obscured by the adapted coordinate expressions in terms of which we defined it, is beyond the scope of this paper and should better be left to the mathematicians. The remarkable connections emerging here between the brane sector of the Batalin-Vilkovisky cohomology of the Hitchin sigma model on one hand and various aspects of Hitchin's generalized complex geometry and Gualtieri's notion of generalized complex submanifold on the other should however be emphasized for its field theoretic and physical mathematical interest.

7. Discussion

The Alexandrov-Kontsevich-Schwartz-Zaboronsky formalism of ref. [52] is a method of constructing solutions of the Batalin-Vilkovisky classical master equation directly, without starting from a classical action with a set of symmetries, as is done in the Batalin-Vilkovisky framework. In ref. [38], using such formalism, Cattaneo and Felder managed to obtain the Batalin-Vilkovisky action of the Poisson sigma model. In ref. [39], the same authors extended their analysis by including branes in the target Poisson manifold and showed that branes had to be coisotropic submanifolds. (See also refs. [53, 54] for a related study.) In spirit, their approach is very close to the one of the present paper. This calls for a comparison of their results and those of we have obtained.

We consider first the case where branes are absent and discuss the incorporation of branes later. Following [38], we view the standard Poisson sigma model as a field theory whose base space, target space and field configuration space are respectively a two dimensional surface Σ, a Poisson manifold M with Poisson bivector P and the space of bundle maps $\phi = (x, y) : \Pi T\Sigma \to \Pi T^*M$.

The supermanifold ΠT^*M has a canonical odd symplectic structure, or P-structure, defined by the canonical odd symplectic form $\omega = du_a dt^a$. With ω, there are associated canonical odd Poisson brackets $(\,,)_\omega$ in standard fashion. The algebra of functions on ΠT^*M with the odd brackets $(\,,)_\omega$ is isomorphic to the algebra of multivector fields on M with the standard Schoutens–Nijenhius brackets.

The space of field configurations inherits an odd symplectic structure from that of ΠT^*M and, so, it also carries a P-structure. The associated odd symplectic form Ω is obtained from ω by integration over $\Pi T\Sigma$ with respect to the usual supermeasure μ (cf.

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eq. (4.5)) and is given by the first term in the right hand side of (5.3). With Ω, there are associated odd Poisson brackets $(\,,)_\Omega$ over the algebra of functions on field configuration space, called Batalin-Vilkovisky antibrackets in the physical literature.

The Poisson bivector field P of M can be identified with a function on ΠT^*M satisfying $(P,P)_\omega = 0$. Its hamiltonian vector field is an odd vector field Q_P such that ${Q_P}^2 = 0$ and defines a so called Q-structure on ΠT^*M.

The Poisson bivector P yields a function S' on field configuration space, again by integration over $\Pi T\Sigma$ with respect to μ, satisfying $(S', S')_\Omega = 0$. S' is given by the second term in the right hand side of (5.5). Its hamiltonian vector field is an odd vector field $Q_{S'}$ such that ${Q_{S'}}^2 = 0$, and, so, it defines a Q-structure on field configuration space.

The base space $\Pi T\Sigma$ carries also a canonical Q-structure d corresponding to the usual de Rham differential on Σ. d induces a Q-structure d on field space in obvious fashion. d is hamiltonian, as indeed $d = Q_{S_0}$, where S_0 is the function on field configuration space defined by the first term in the right hand side of (5.5). S_0 satisfies $(S_0, S_0)_\Omega = 0$.

One verifies that $(S', S_0)_\Omega = 0$. The sum $S = S_0 + S'$ thus satisfies $(S, S)_\Omega = 0$. S is nothing but the Batalin-Vilkovisky action of the Poisson sigma model satisfying the Batalin-Vilkovisky master equation. Its hamiltonian vector field $\delta_{BV} = Q_S$ is the Batalin-Vilkovisky variation operator.

In the untwisted case $H = 0$, the construction of the Hitchin sigma model of ref. [31] closely parallels that of the Poisson sigma model outlined above. The field space is the same as that of the Poisson sigma model. The field space odd symplectic form Ω is also the same, being derived from the same odd symplectic form ω of ΠT^*M exactly in the same way. The relevant difference is that the role of the Poisson bivector P is played here by the generalized complex structure $\mathcal{J}$, which does not correspond to any multivector in general, but which still corresponds to some structure of target space. The overall framework is otherwise totally analogous.

In the twisted case $H \neq 0$, the situation is more complicated. If $H = d_M B$ is exact, we can b transform H to 0 by setting $b = B$ (cf. eq. (2.4)), recovering in this way the untwisted case, and follow the lines outlined above. If H is not exact, this is no longer possible. The symplectic form Ω given by (5.5) cannot be straightforwardly obtained from a corresponding structure of ΠT^*M. Further, the action functionals contains a topological Wess-Zumino term and so it involves an extra geometrical datum, the extension of the embedding field x to a 3-dimensional manifold Γ bounded by Σ. The spirit of the Alexandrov-Kontsevich-Schwartz-Zaboronsky formalism, however, is still fully discernible, in spite of the technical differences.

Let us now examine the case when branes are present. In ref. [39], the Poisson sigma model in the presence of branes is studied. It is found that branes are coisotropic submanifolds W of the target Poisson manifold (M, P). For a given brane W, the boundary conditions impose that the world sheet field ϕ maps $\Pi T\partial\Sigma$ into the parity reversed conormal bundle ΠN^*W. Writting $\phi = (x, y)$ as usual and using adapted coordinates, the conditions are given by (5.17a), (5.17b) with $F_{ij} = 0$.

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In this work, we study similarly the Hitchin sigma model in the presence of branes. We find that branes are generalized complex submanifolds (W, F) of the target twisted generalized complex manifold $(M, H, \mathcal{J})$ (cf. section 3). For a given (W, F), the boundary conditions impose that the world sheet fields ϕ maps $\Pi T\partial\Sigma$ into the the parity reversed generalized tangent bundle $\Pi\mathcal{T}^F W$ of W (cf. sections 3 and 5). Writting $\phi = (x, y)$ and using adapted coordinates again, the boundary conditions are given by (5.17a), (5.17b) and (5.25).

Now, we recall that not all Poisson manifolds (M, P) are particular cases of twisted generalized complex manifolds $(M, H, \mathcal{J})$. Only those with pointwise invertible P are. For these, $H = 0$ and $\mathcal{J}$ is of the form (2.22) with $Q = -P^{-1}$. For this reason, the Hitchin sigma model is only a partial generalization of the Poisson sigma model since it can reproduce the latter only in the case where the target space Poisson bivector is pointwise invertible [31].

Similarly, when (M, P) is a Poisson manifold with pointwise invertible P, not all coisotropic submanifolds W of M are generalized complex submanifolds (W, F) for some F of the corresponding generalized complex manifold $(M, H = 0, \mathcal{J})$ with $\mathcal{J}$ of the form indicateded above. If we require, as it is reasonable to do, that $F = 0$, then W must be a lagrangian submanifold of M in order to be generalized complex submanifold, a more restrictive condition than being a coisotropic submanifold. This follows easily from (3.9a, (3.9c), by inspection. So, again, the brane Hitchin sigma model is only a partial generalization of the brane Poisson sigma model.

As to the boundary conditions, the conditions (5.17a), (5.17b) with $F_{ij} = 0$ reproduce those of ref. [39] as already noticed. We have, however, the extra condition (5.25) with $J^i{}_j = 0$ and $F_{ij} = 0$. This condition follows from the requirement that the boundary integral (5.23) vanishes. Of course, when $F = 0$, this condition is no longer strictly necessary.

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